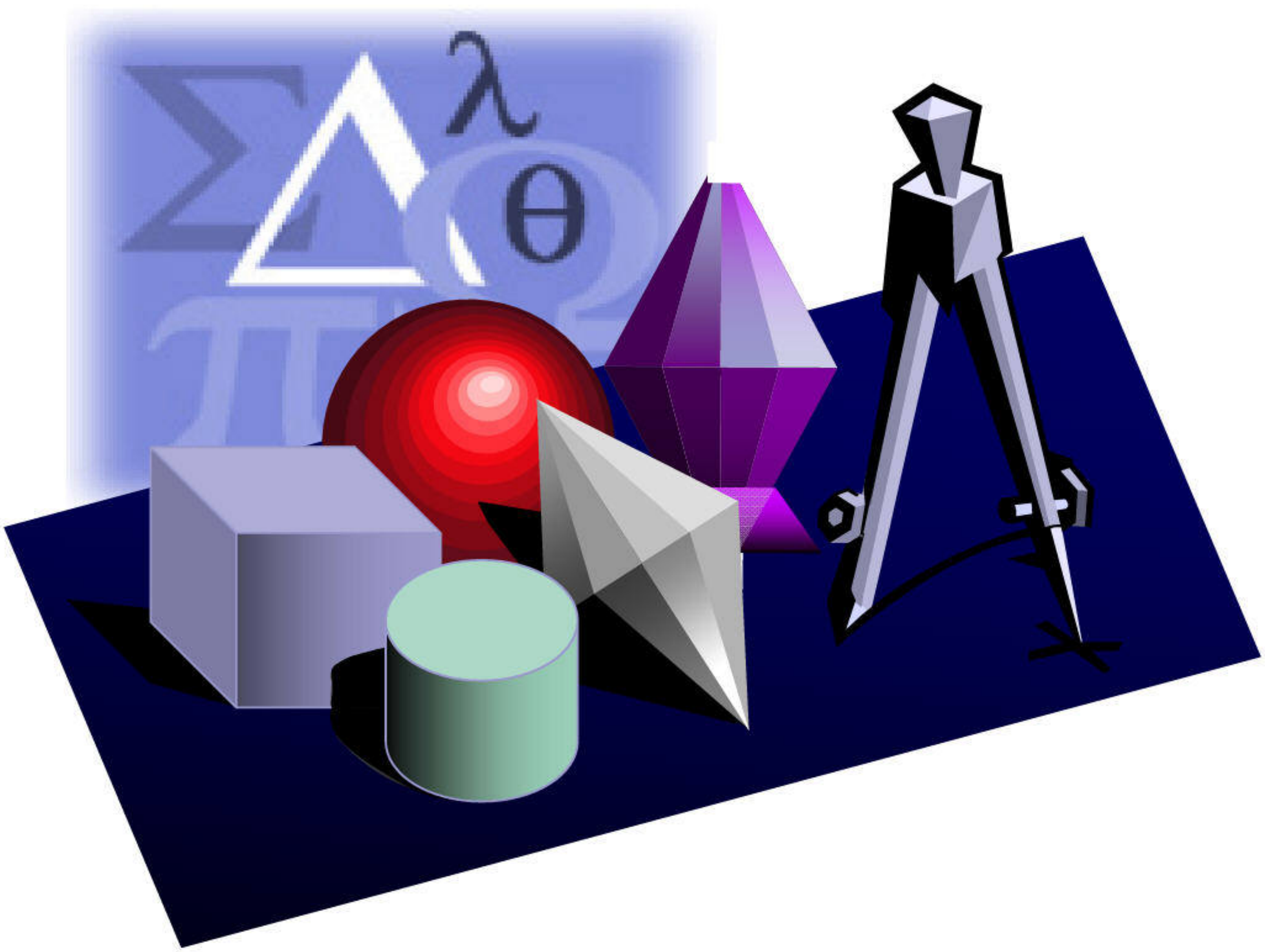
Salisbury University Department of Physics

Calculus & Calculating Moment of Inertia



SU PHYSICS

The Moment of Inertia of Rigid Masses

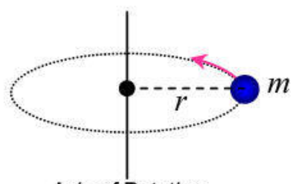
Moment of Inertia:

The moment of inertia of an object about an axis relates the angular velocity of an object (how fast it's spinning) to how much angular momentum it has. This tells us how much torque must be exerted on the object to get it spinning, or how much energy it has when it's spinning at some rate. To determine the rotational inertia of an arbitrarily shaped object generally involves a thorough understanding of calculus techniques and geometry because a rotating geometrical solid has a continuous distribution of mass at a continually varying distance from the rotation axis. The moment of inertia of a rigid body can be determined using the following steps using whatever coordinate system may be convenient for the calculation at hand depending on the rotational axis and symmetry of the solid object.

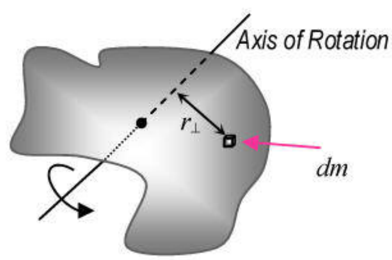
STEP ONE:

Realizing that the rotational inertia (I) of a point mass (m) revolving around an axis at a distance (r) from that axis is given by;

$$I = mr^2$$



we can break up any object into a large collection of infinitesimally small masses (dm) each with their own radial distances from the rotation axis. This kind of mass element is called a differential element. We will need to find the moment of inertia for each individual mass element (dm) about the axis.



Each element rotating about the axis will behave as an individual point mass with an infinitesimally small moment of inertia (dI) given by;

$$dI = r_\perp^{\,2} dm$$

Note that the differential element of moment of inertia (dI) must always be defined with respect to a specific rotation axis.

Step Two

The moment of inertia of the body is then the sum of all these differential elements. This "integration" of the individual small pieces creates the whole body that is being calculated. This is "integrating" the individual pieces or "integration of the mathematical" function over the extent of the geometrical rigid rotating body.

The sum over all these mass elements is called an integral over the mass and will determine the moment of inertia.

$$I = \int dI = \int_{object} r_{\perp}^{2} dm$$

If the rigid body is nearly a one dimensional object (say a thin long rod) then this integral is in one dimension. Likewise, if the rigid body is planar (a disk shape) it is a double integral over two dimensions and if the mass is a three dimensional object (a sphere, for example) then it will be a triple integral. Picking symmetric shapes and/or different coordinate systems to tackle the integration can make the mathematical work much easier as you will be able to see in the calculations.

Step Three

In some cases, it may also be necessary to use the Parallel Axis Theorem to fully determine the moment of inertia about more exotic rotation axis that a rigid body may be revolving.

Parallel Axis Theorem: The moment of inertia of a body about any given axis is the moment of inertia about a parallel axis through the center of mass, plus the moment of inertia about the given axis if all the mass of the body were located at the center of mass.

$$I = I_{cm} + md^{2}$$

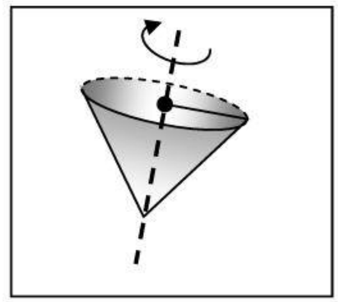
What follows in this section is the application of these physical principles and calculus techniques to a set of geometrical solids often found as the core shapes in bowling balls. We are able to theoretically calculate the shaped core moments of inertia and compare them to some experimental measures and to literature while investigating the fundamental issues of the "shape" and its motional dynamics.

Theoretical Moment of Inertia: Right Solid Cone

Result:

Cone: z-axis rotation

$$I_{z-axis} = \frac{3}{10} MR^2$$



Mathematical Derivation

$$I = \int_0^M r^2\,dm \qquad dm = \rho dV \qquad \text{Assuming constant density}$$

$$I = \int dI \quad = \int I_{disk}\,dx = \int \frac{1}{2} M_{disk} (r')^2\,dz$$

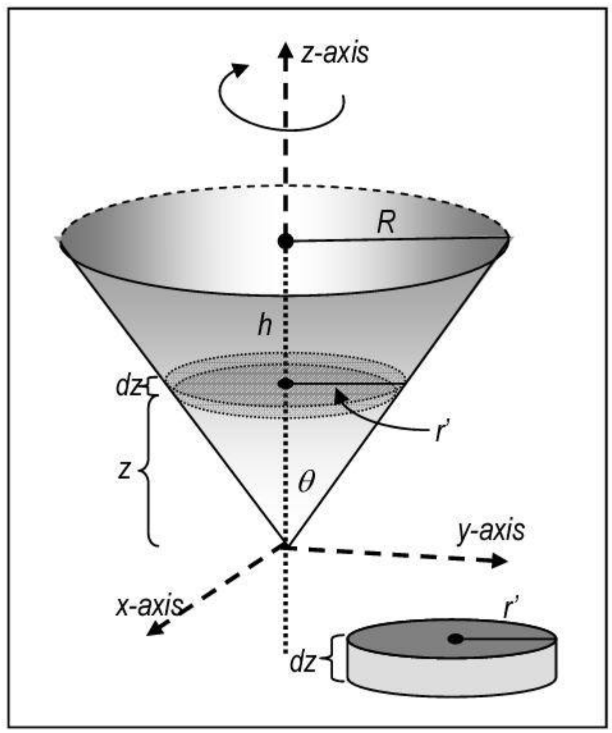
$$M_{disk} = \rho(\text{Volume}) \;\Rightarrow\; dm = \rho dV = \rho A dz$$

r' is the radius of a disk element at some "z" value from the origin.

$$\frac{R}{h} = \tan\theta = \frac{r'}{z} \quad \Rightarrow \quad r' = \left(\frac{R}{h}\right) z$$

$$dm = \rho A dz = \rho\left(\pi (r')^2\right) dz$$

Putting all this together, using each "disk" element of the cone up from the vertex along the z-axis, we get the following;



$$I_{cone} = \int \frac{1}{2}\left[\rho(A)\right]\left[(r')^2\right] dz$$

$$I_{cone} = \int \frac{1}{2}\rho\left(\pi (r')^2\right)\left[(r')^2\right] dz$$

Now, plug in how r' varies with z

$$I_{cone} = \int_0^h \frac{1}{2}\rho\left(\pi\left(\left(\frac{R}{h}\right)z\right)^2\right)\left[\left(\left(\frac{R}{h}\right)z\right)^2\right] dz$$

$$I_{cone} = \frac{1}{2}\rho\pi\int_0^h \left(\frac{R}{h}z\right)^2\left(\frac{R}{h}z\right)^2 dz$$

$$I_{cone} = \frac{1}{2}\rho\pi\int_0^h \left(\frac{R^4}{h^4}z^4\right) dz$$

$$I_{cone} = \frac{1}{2}\rho\pi\frac{R^4}{h^4}\int_0^h \left(z^4\right) dz = \frac{1}{2}\rho\pi\frac{R^4}{h^4}\left(\frac{z^5}{5}\right)\Bigg|_0^h$$

$$I_{cone} = \frac{1}{2}\rho\pi\frac{R^4}{h^4}\frac{h^5}{5} = \frac{1}{10}\rho\pi R^4 h$$

Assuming a uniform density;

$$\rho = \frac{\text{Mass}}{\text{Volume}} = \frac{M}{\frac{1}{3}\pi R^2 h} \quad \text{then...} \quad I = \frac{1}{10}\left(\frac{M}{\frac{1}{3}\pi R^2 h}\right)\pi R^4 h$$

then $\Rightarrow$ $$I_{cone} = \frac{3}{10}MR^2$$

Mathematical Derivation

$$I_{cone} = \int_0^h \frac{1}{4}\rho\left(\pi\left(\left(\frac{R}{h}\right)x\right)^2\right)\left[\left(\left(\frac{R}{h}\right)x\right)^2\right]dx + \int_0^h \rho\pi\left(\left(\frac{R}{h}\right)x\right)^2 x^2\, dx$$

$$I_{cone} = \frac{1}{4}\rho\pi\int_0^h\left(\frac{R}{h}x\right)^2\left(\frac{R}{h}x\right)^2 dx + \rho\pi\int_0^h\left(\left(\frac{R}{h}\right)x\right)^2 x^2\, dx$$

$$I_{cone} = \frac{1}{4}\rho\pi\int_0^h\left(\frac{R^4}{h^4}x^4\right)dx + \rho\pi\int_0^h\left(\frac{R^2}{h^2}\right)x^4 dx$$

$$I_{cone} = \frac{1}{4}\rho\pi\frac{R^4}{h^4}\int_0^h\left(x^4\right)dx + \rho\pi\frac{R^2}{h^2}\int_0^h\left(x^4\right)dx$$

$$I_{cone} = \frac{1}{4}\rho\pi\frac{R^4}{h^4}\left(\frac{x^5}{5}\right)\Bigg|_0^h + \rho\pi\frac{R^2}{h^2}\left(\frac{x^5}{5}\right)\Bigg|_0^h$$

$$I_{cone} = \frac{1}{4}\rho\pi\frac{R^4}{h^4}\frac{h^5}{5} + \rho\pi\frac{R^2}{h^2}\frac{h^5}{5}$$

$$I_{cone} = \frac{1}{20}\rho\pi R^4 h + \frac{1}{5}\rho\pi R^2 h^3$$

Assuming a uniform density;

$$\rho = \frac{\text{Mass}}{\text{Volume}} = \frac{M}{\frac{1}{3}\pi R^2 h}$$

then...

$$I = \frac{1}{20}\left(\frac{M}{\frac{1}{3}\pi R^2 h}\right)\pi R^4 h + \frac{1}{5}\left(\frac{M}{\frac{1}{3}\pi R^2 h}\right)\pi R^2 h^3$$

then $\Rightarrow$ $\boxed{I_{cone} = \frac{3}{20}MR^2 + \frac{3}{5}Mh^2}$

Now, using the parallel-axis thereom, where any moment inertia can be found from;

$I = I_{\text{center of mass}} + md^2$ where d is the distance from the center of mass axis to the new axis, we can find the following;

$$I_{\text{center of mass}} = \frac{3}{20}MR^2 + \frac{3}{5}Mh^2 - M\left(\frac{3}{4}h\right)^2$$

$\boxed{I_{cm} = \frac{3}{20}MR^2 + \frac{3}{80}Mh^2}$ This will be around an axis through the center of mass & parallel to the cone's base. Also, we can get

$$I_{base} = I_{cm} + md^2 = \frac{3}{20}MR^2 + \frac{3}{80}Mh^2 + M\left(\frac{1}{4}h\right)^2$$

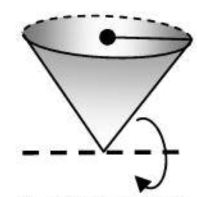
$\boxed{I_{base} = \frac{3}{20}MR^2 + \frac{1}{10}Mh^2}$ This will be around an axis parallel and through the base of a cone.

Results:

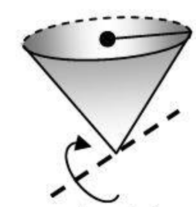
Cone: x & y axis rotation

$$I_{x-axis} = I_{y-axis}$$

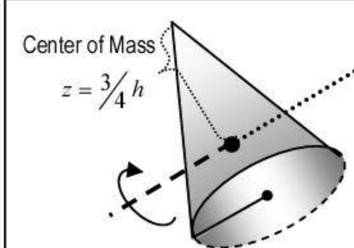
$$I_x = I_y = \frac{3}{20}MR^2 + \frac{3}{5}Mh^2$$



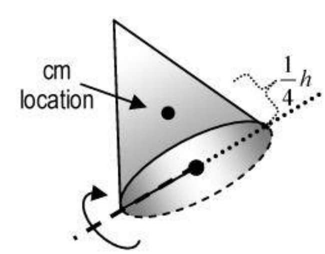
y-axis rotation



x-axis rotation



cm-axis rotation



base-axis rotation

Mathematical Derivation

Using symmetry and the previous results from the single cone calculations, we find the following;

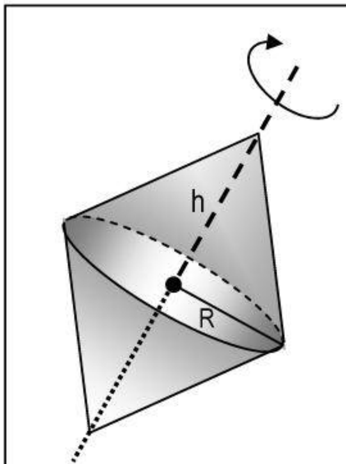
$$I_{base-axis} = 2\left(\frac{3}{20}MR^2 + \frac{1}{10}Mh^2\right)$$

$$I_{base-axis} = \frac{3}{10}MR^2 + \frac{1}{5}Mh^2$$

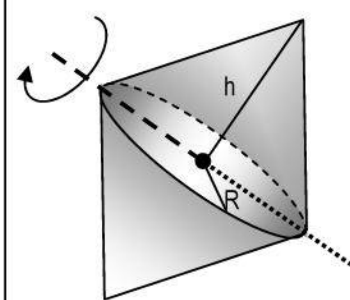
Likewise,

$$I_{central-axis} = 2\left(\frac{3}{10}MR^2\right)$$

$$I_{central-axis} = \frac{3}{5}MR^2$$



central-axis



base-axis

Mathematical Derivation

$$I = \int_0^M r^2 dm \qquad dm = \rho dV \qquad \text{Assuming constant density}$$

$$I = \int dI = \int r^2 \rho dV = \rho \int r^2 dV \qquad r = \sqrt{x^2 + y^2}$$

Now since there are four quadrants to sum:

$$I = 4 * \rho \int_0^{A/2} \int_0^{B/2} \int_0^{C} (x^2 + y^2) dx\, dy\, dz$$

$$I = 4\rho \int_0^{B/2} \int_0^{C} \left(\frac{x^3}{3} + y^2 x \right) \Bigg|_0^{A/2} dy\, dz$$

$$I = 4\rho \int_0^{B/2} \int_0^{C} \left(\frac{A^3}{24} + \frac{y^2 A}{2} \right) dy\, dz$$

$$I = 4\rho \int_0^{C} \left(\frac{A^3 y}{24} + \frac{y^3 A}{6} \right) \Bigg|_0^{B/2} dz$$

$$I = 4\rho \int_0^{C} \left(\frac{A^3 B}{48} + \frac{B^3 A}{48} \right) dz$$

$$I = 4\rho \left(\frac{A^3 Bz}{48} + \frac{B^3 Az}{48} \right) \Bigg|_0^{C}$$

$$I = 4\rho \left(\frac{A^3 BC}{48} + \frac{B^3 AC}{48} \right)$$

$$I = \frac{\rho ABC}{12} \left(A^2 + B^2 \right)$$

but $\rho = \frac{\text{mass}}{\text{volume}} = \frac{M}{ABC}$

$$I = \frac{M}{12} \left(A^2 + B^2 \right)$$

and for Solid Cube (A = B = C = S)

$$I_{cube} = \frac{M}{6} \left(S^2 \right)$$

I = moment of inertia
M = mass of core
A, B, C are side lengths
S = side length of cube

Results:

Rectangular Solid

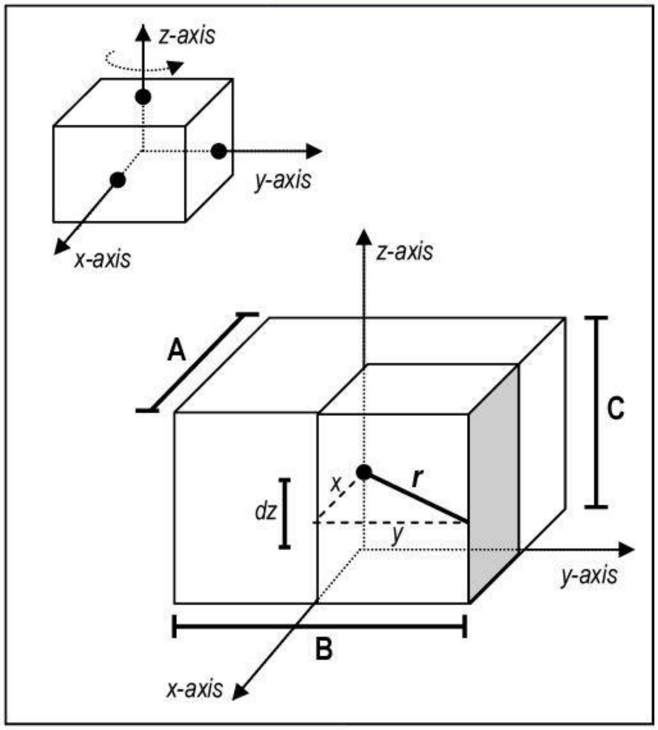
$$I_z = \frac{M}{12} \left(A^2 + B^2 \right)$$

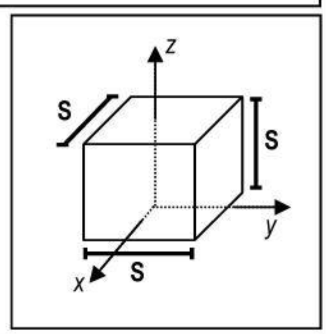
$$I_x = \frac{M}{12} \left(B^2 + C^2 \right)$$

$$I_y = \frac{M}{12} \left(A^2 + C^2 \right)$$

Result:

Solid Cube





Mathematical Derivation

Result

Solid Sphere

$$I = \int_0^M r^2\,dm \qquad dm = \rho dV \qquad \text{Assuming constant density}$$

$$I = \int dI \quad = \int \frac{1}{2} y^2 \rho dV \quad = \frac{1}{2}\rho \int y^2 \pi\, y^2 dz$$

$$I = \frac{1}{2}\pi\rho \int_{-R}^{R} y^4 dz \qquad \text{with} \qquad y = \sqrt{R^2 - z^2}$$

so

$$I = \frac{1}{2}\pi\rho \int_{-R}^{R} \left(R^2 - z^2\right)^2 dz$$

$$I = \frac{1}{2}\pi\rho \int_{-R}^{R} \left(R^4 - 2R^2 z^2 + z^4\right) dz$$

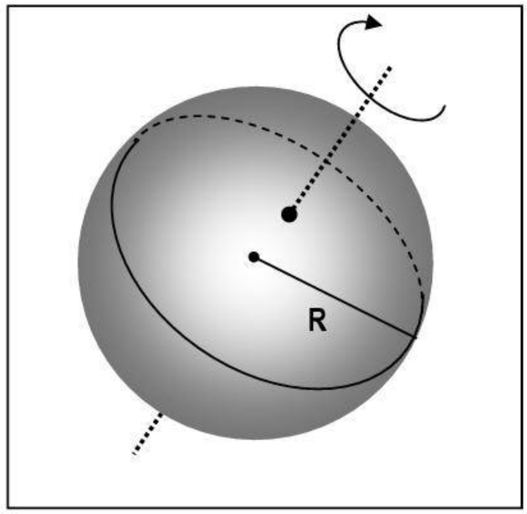
$$I = \frac{1}{2}\pi\rho \left(R^4 z - \frac{2R^2 z^3}{3} + \frac{z^5}{5} \right)\Bigg|_{-R}^{R}$$

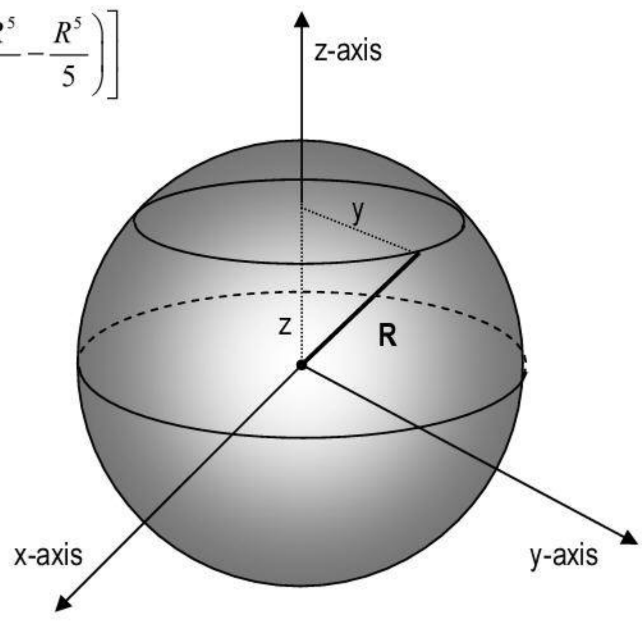
$$I = \frac{1}{2}\pi\rho \left[\left(R^5 - \frac{2R^5}{3} + \frac{R^5}{5} \right) - \left(-R^5 + \frac{2R^5}{3} - \frac{R^5}{5} \right)\right]$$

$$I = \frac{1}{2}\pi\rho \left[\left(2R^5 - \frac{4R^5}{3} + \frac{2R^5}{5} \right)\right]$$

$$I = \frac{1}{2}\pi\rho \left(\frac{16}{15} R^5 \right)$$

$$I = \frac{8}{15}\pi\rho R^5$$





Assuming a uniform density:

$$\rho = \frac{\text{Mass}}{\text{Volume}} = \frac{M}{\frac{4}{3}\pi R^3}$$

then...

$$I = \frac{8}{15}\pi \left(\frac{M}{\frac{4}{3}\pi R^3} \right) R^5 \quad \Rightarrow \quad I_{sphere} = \frac{2}{5} M R^2$$

Mathematical Derivation

$$I = \int_0^M r^2 dm \qquad dm = \rho dV$$ Assuming constant density

$$dV = L dA = L(2\pi r)dr$$

$$I = \int dI = \int r^2 \rho (2\pi r L) dr$$

$$I = \int_0^R r^3 \rho (2\pi L) dr$$

$$I = \rho (2\pi L) \int_0^R r^3 dr$$

$$I = \rho (2\pi L) \left(\frac{r^4}{4} \right) \Bigg|_0^R$$

$$I = 2\rho\pi L \left(\frac{R^4}{4} \right)$$

$$I = \frac{1}{2} \rho \pi L R^4$$

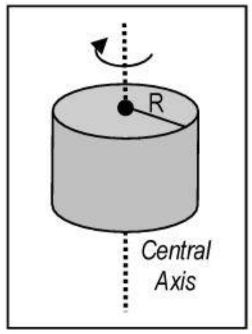
Assuming a uniform density:

$$\rho = \frac{\text{Mass}}{\text{Volume}} = \frac{M}{\pi R^2 L}$$

then...

$$I = \frac{1}{2} \left(\frac{M}{\pi R^2 L} \right) \pi L R^4$$

then $\Rightarrow$ $$I_{cylinder} = \frac{1}{2} MR^2$$



Results

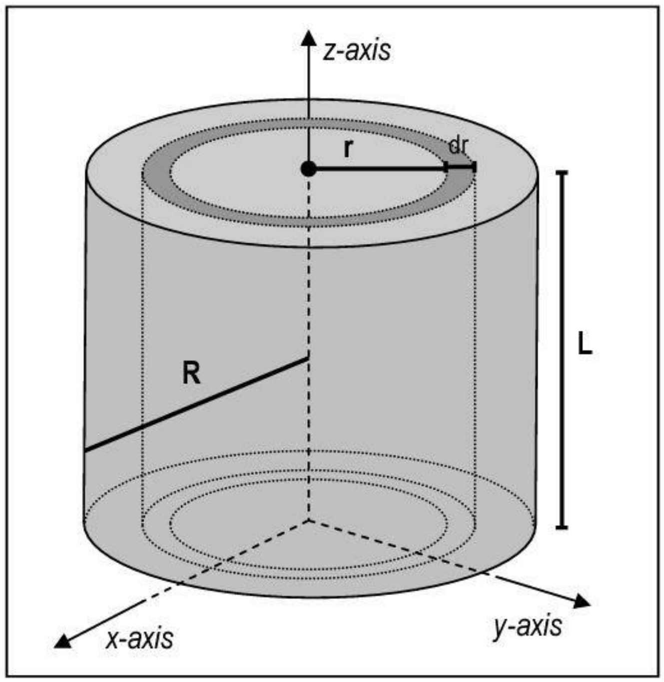
Central Axis

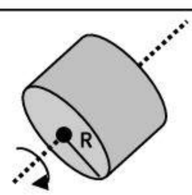
$$I_{cylinder} = \frac{1}{2} MR^2$$

Central Diameter

$$I = \frac{1}{4} MR^2 + \frac{1}{12} ML^2$$

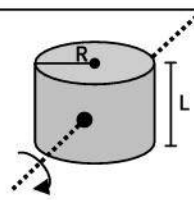
End Diameter





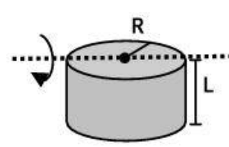
Central Axis

$$I = \frac{1}{2} MR^2$$



Central Diameter

$$I = \frac{1}{4} MR^2 + \frac{1}{12} ML^2$$



End Diameter

$$I = \frac{1}{4} MR^2 + \frac{1}{3} ML^2$$

Mathematical Derivation

$I = \int_0^M R^2 dm \qquad dm = \rho dV$ Assuming constant density

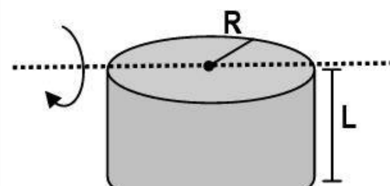
$I = \int dI \quad = \int R^2 \rho dV$

$$I_{y-axis} = \int_0^L \frac{1}{4}\rho\left(\pi R^2\right)R^2 dz + \int_0^L \rho\left(\pi R^2\right)z^2 dz$$

$$I_{y-axis} = \frac{1}{4}\rho\pi R^4 \int_0^L dz + \rho\pi R^2 \int_0^L z^2 dz$$

$$I_{y-axis} = \frac{1}{4}\rho\pi R^4 \left(z\right)\Big|_0^L + \rho\pi R^2 \left(\frac{z^3}{3}\right)\Bigg|_0^L$$

$$I_{y-axis} = \frac{1}{4}\rho\pi R^4 L + \rho\pi R^2 \left(\frac{L^3}{3}\right)$$



End Diameter

$I = \frac{1}{4}MR^2 + \frac{1}{3}ML^2$

Assuming a uniform density;

$$\rho = \frac{\text{Mass}}{\text{Volume}} = \frac{M}{\pi R^2 L}$$

then...

$$I = \frac{1}{4}\left(\frac{M}{\pi R^2 L}\right)\pi R^4 L + \left(\frac{M}{\pi R^2 L}\right)\pi R^2 \left(\frac{L^3}{3}\right)$$

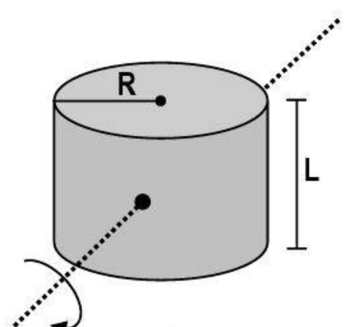
then $\Rightarrow$ $\boxed{I_{y-axis} = \frac{1}{4}MR^2 + \frac{1}{3}ML^2}$

Now, using the parallel-axis thereom, where any moment inertia can be found from;

$I = I_{\text{center of mass}} + md^2$ where d is the distance from the center of mass axis to the new axis, we can find the following;

$$I_{\text{center of mass}} = \frac{1}{4}MR^2 + \frac{1}{3}ML^2 - M\left(\frac{1}{2}L\right)^2$$

$\boxed{I_{cm} = \frac{1}{4}MR^2 + \frac{1}{12}ML^2}$ This will be around an axis through the center of mass & parallel to the cylinders's base.

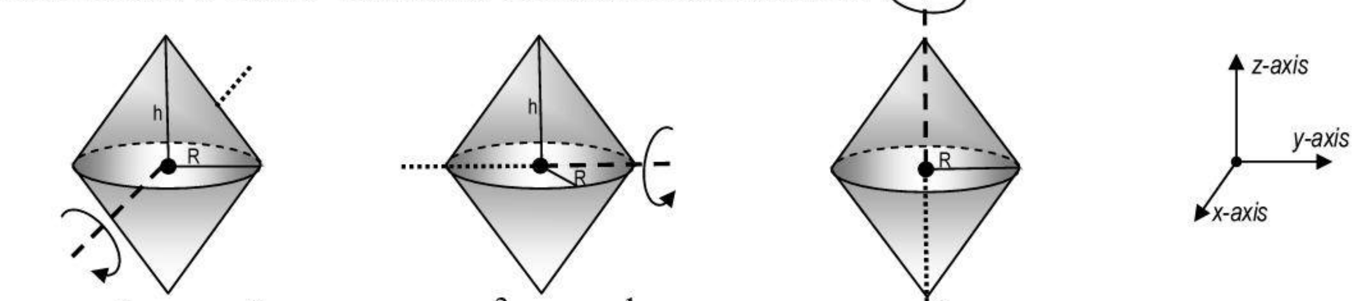


Central Diameter

$I = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$

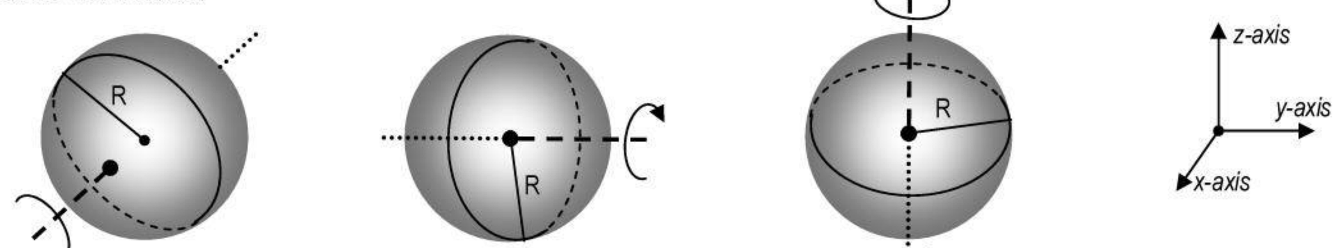
Theoretical Summary Table

Solid Double Cone–Diamond Core Approximation



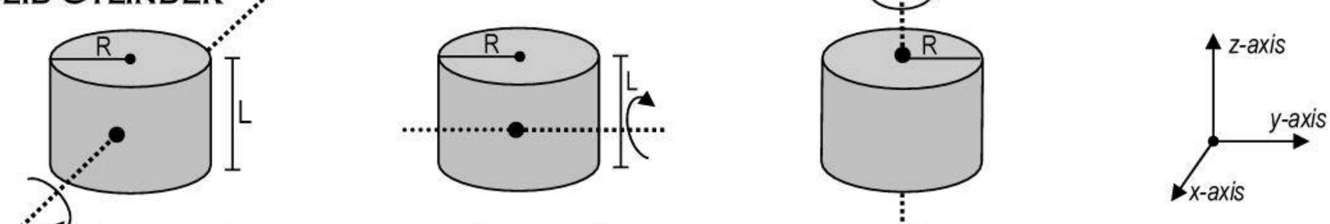
$$I_{x-axis} = \frac{3}{10}MR^2 + \frac{1}{5}Mh^2 \qquad I_{y-axis} = \frac{3}{10}MR^2 + \frac{1}{5}Mh^2 \qquad I_{z-axis} = \frac{3}{5}MR^2$$

Solid Sphere



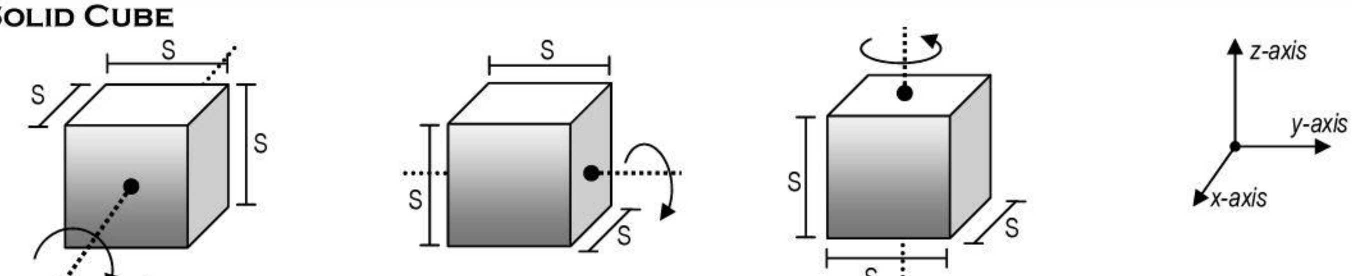
$$I_{x-axis} = \frac{2}{5}MR^2 \qquad I_{y-axis} = \frac{2}{5}MR^2 \qquad I_{z-axis} = \frac{2}{5}MR^2$$

Solid Cylinder



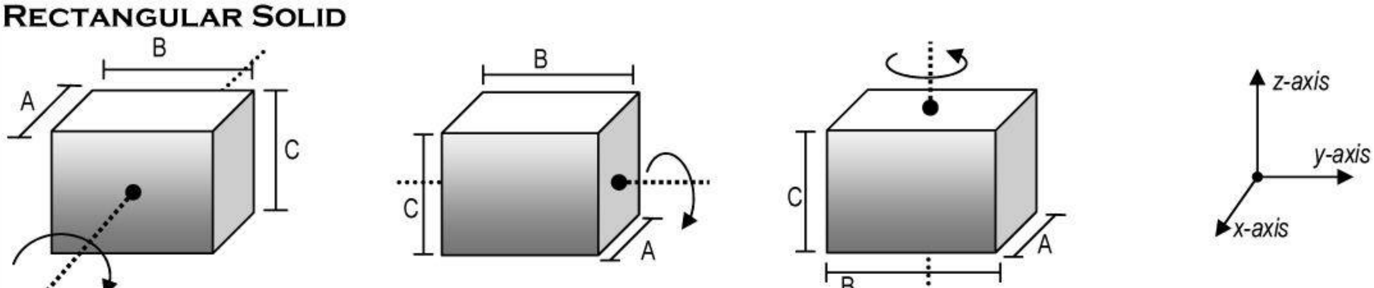
$$I_{x-axis} = \frac{1}{4}MR^2 + \frac{1}{12}ML^2 \qquad I_{y-axis} = \frac{1}{4}MR^2 + \frac{1}{12}ML^2 \qquad I_{x-axis} = \frac{1}{2}MR^2$$

Solid Cube



$$I_{x-axis} = \frac{1}{6}MS^2 \qquad I_{y-axis} = \frac{1}{6}MS^2 \qquad I_{z-axis} = \frac{1}{6}MS^2$$

Rectangular Solid



$$I_{x-axis} = \frac{1}{12}M\left(B^2 + C^2\right) \qquad I_{y-axis} = \frac{1}{12}M\left(A^2 + C^2\right) \qquad I_{z-axis} = \frac{1}{12}M\left(A^2 + B^2\right)$$